ELEMENTARY THEORY OF CONCENTRATED DISPERSED SYSTEMS

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A closed system of equations is derived for the energy flux, and the boundary conditions are given. The transport coefficients and other parameters are found from elementary gaskinetic considerations for a high concentration of the solid phase. As an example, the solution is found for the problem of an "adiabatic" Couette flow for a granulated medium.

1. The hydrodynamics of multiphase systems does not yet have a satisfactory theoretical basis despite the fact that it is of wide practical application and that a very considerable amount of experimental data has been accumulated [1]. The averaged equations of motion of two-phase streams are not closed and can be applied to nonuniform flows provided that the dimensions of the elements in the dispersed phase are much smaller than the characteristic size of the channel or boundary layer.

The equations of averaged motion of the phases can be obtained in closed form if the interaction mechanism is known. Two limiting situations can be distinguished: the concentration of the dispersed phase is small, and its elements interact only with the carrier medium (gas, liquid); the concentration of the dispersed phase is close to maximum, and the movements of the elements have an order which is significantly smaller than their characteristic size.

It is the latter type of system that we consider in this paper (a generalization of [2]). We take a two-fluid model described by the phenomenological hydrodynamic equations in which the transport coefficients and other quantities are derived from elementary kinetic considerations. These considerations are approximate but they do allow the principal relations to be understood and do reflect the main properties of the various phenomena.

We follow a method in which the two phases are described separately and their interaction is taken into account by means of a body force F_i. Since the size of the solid particles is taken to be much smaller than the characteristic dimensions of the flow and since their concentration is considerable, the gradient of the "averaged" velocity of the fluid is negligibly small in comparison with the local gradients near the particles and so the equations of motion of the fluid can be written in the Euler form

$$\rho\varepsilon \frac{dv_i}{dt} = -F_i - \frac{\partial p}{\partial x_i} + \rho\varepsilon g_i \tag{1.1}$$

$$\frac{\partial(\rho\varepsilon)}{\partial t} + \frac{\partial(\rho\varepsilon v_i)}{\partial x_i} = 0 \tag{1.2}$$

The force F_i which acts on the stream from the particles is greater than the Stokes term in order of magnitude. Here ρ is the density of the fluid, ε is the porosity (fraction of the fluid in unit volume), p is the pressure of the fluid phase, g_i is the acceleration due to mechanical forces, t is the time, and v_i is the average velocity of the fluid, which is related to the velocity in an empty cross section v_{0i} by the equation $v_{0i} = \varepsilon v_i$.

For the solid phase, the equations of motion are

$$\rho_s\tau \frac{dw_i}{dt} = F_i + \frac{\partial T_{ij}}{\partial x_j} + \rho_s\tau g_i \tag{1.3}$$

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$$\frac{\partial(\rho_s\tau)}{\partial t} + \frac{\partial(\rho_s\tau w_i)}{\partial x_i} = 0 \quad (1.4)$$

where ρ_s is the density of the solid phase material, $\tau = 1 - \varepsilon$ is the volume concentration of the solid phase, w_i is the average flow speed of this phase, and T_{ij} is the stress tensor in the solid phase considered as a continuous medium and reflects the interaction of the solid phase particles.

We consider the solid phase to consist of identical spheres of diameter d. The number of particles in unit volume $n = 6\tau/\pi d^3$. Of the forces acting on a particle from the fluid phase, we consider the hydraulic resistance and the transverse Magnus force which acts for a flow round a rotating sphere. We can thus write

$$f_i = \zeta\frac{\pi d^2}{4}\frac{\rho|\mathbf{u}|}{2}u_i - \frac{\pi d^3}{6}\frac{\partial p}{\partial x_i} + \frac{\pi d^3}{3}\rho[\bar{\mathbf{u}}, \boldsymbol{\omega}]_i \quad (1.5)$$

Here ζ is the hydraulic resistance coefficient of a particle, ω is the angular velocity vector, $u_i = \varepsilon\psi^{-1}(v_i - w_i)$ is the maximum flow velocity past the particle as defined by the minimum relative transfer cross section ψ. The expression for the Magnus force has been obtained in [3], and $\psi = 1 - 1.17\,\tau^{2/3}$ [2], $\bar{u}_i = \varepsilon\varphi^{-1}(v_i - w_i)$ is the average flow velocity round a particle. The approximate relationship $\varphi = 1.09 \cdot (1 - J^{2/3})$ can be obtained by a method similar to that used in [2] for calculating ψ.

We assume in the calculation of F_i that the vector ω for the particles in unit volume is randomly directed and that therefore the Magnus force, being an internal one, does not enter the momentum equations. Thus

$$F_i = n\langle f_i\rangle = \frac{3}{4}\frac{\zeta\tau}{d}\rho|\mathbf{u}|u_i - \tau\frac{dp}{\partial x_i} \quad (1.6)$$

Equation (1.1) now becomes

$$\rho\frac{dv_i}{dt} = -\frac{\partial p}{\partial x_i} - \frac{3}{4}\frac{\zeta\tau\rho}{\varepsilon d}|\mathbf{u}|u_i \quad (1.7)$$

By comparing (1.7) and (1.1), we can decide about the necessity of including the factor ε in front of $\partial p/\partial x_i$.

In order to determine the tensor T_{ij}, we assume that the system of particles can be considered as a fluid which satisfies the Stokes postulates [4]. We can then write for T_{ij} the general expression [4]

$$T_{ij} = -p_s^\circ\delta_{ij} + \mu D_{ij} - \gamma D_{ik}D_{kj} \quad (D_{ij} = \partial w_i/\partial x_j + \partial w_j/\partial x_i) \quad (1.8)$$

where D_{ij} is the strain rate tensor, p°_s, μ, γ, are scalars which depend on the thermodynamic parameters and the invariants of D_{ij}. We introduce the quantity $p_s = -\frac{1}{3}(T_{ii} + \gamma D_{ik}^2)$, which represent the analog of the hydrostatic pressure for a system of particles. Expression (1.8) becomes

$$T_{ij} = -\left(p_s + \frac{2}{3}\mu\frac{\partial w_k}{\partial x_k}\right)\delta_{ij} + \mu D_{ij} - \gamma D_{ik}D_{kj} \quad (1.9)$$

We assume that as with a normal gas the system has no "memory," i.e., we neglect such effects as second viscosity. This is justified by the fact that the particles do not have internal degrees of freedom except rotational ones and these are considered to be rapidly relaxing.

In order to determine the quantities p_s, μ, and γ, we take a more definite model of the medium and consider the solid phase as a gas of solid spheres. The distance between particles τ was found in [2] as $l = d[\tau_0/\tau)^{1/3} - 1]$ on the assumption of similarity between nondense packing with density τ and some "standard" dense packing of density τ_0.

The particles in the dispersion medium have some "average" motion and also possess random velocities. For a sufficiently dense medium with $l < d$, a particle can only collide with nearest neighbors and the remaining particles are inaccessible (the phenomenon of "screening"). The mean free path of a particle $\sim 2l$ and the time between collisions $t \approx 2l/c\sqrt{3}$, where c is the random velocity of the particles appropriate to one degree of freedom.

The collision time of two spheres is determined by the speed of sound c^* in the material of the spheres and is equal in order of magnitude to $t^* \approx 2d/c^*$; thus the fraction of spheres in a collision state is

(for $c \ll c^*$)

$$\beta_2 = \frac{t^*}{t} = \sqrt{3}\,\frac{d}{l}\,\frac{c}{c^*}$$

and the fraction of spheres in third- or higher-order collisions is given in order of magnitude by

$$\beta_3 \approx \frac{\beta_2}{2}\,2\beta_2 3 = 3\beta_2{}^2 = \frac{9\,(c/c^*)^2}{[(\tau_0/\tau)^{1/3} - 1]^2} \tag{1.10}$$

The use of the gas model presupposes that

$$\beta_3 \lll 1 \tag{1.11}$$

Thus multiple collisions, and hence collective effects, can be neglected.

By concentrated systems we mean those for which

$$l\,/\,d = (\tau_0\,/\,\tau)^{1/3} - 1 < 1 \tag{1.12}$$

(with $\tau > 0.075$).

Because of the high value of c^*, there is a wide range of concentrations for which (1.11) and (1.12) are satisfied simultaneously. Further analysis will refer to this region and, in particular cases, to the region $l/d \geq 1$.

Because of the screening effect, the kinetic coefficients can be calculated from a model consisting simply of two layers of particles situated at a distance $l + d$ from each other. The first layer can be taken as fixed and the second as moving with an average velocity Δw. The particles in the layers also have a vertical velocity c connected with the random motion.

The number of particles entering unit area of a layer

$$n(l + d) = \frac{6\tau}{\pi d^2}\left(\frac{\tau}{\tau_0}\right)^{1/3}$$

An error is made in [2] at this point but it has little effect on the quantitative results. One sphere of mass m undergoes $c/2l$ collisions in unit time. The total number of spheres crossing unit area in unit time is

$$N = 3\,c\tau\,/\,\pi d^3\eta, \qquad \eta = 1 - (\tau\,/\,\tau_0)^{1/3}$$

so that the momentum transmitted normally through unit area of surface in unit time is

$$p_s = 2\,mc\,N = \rho_s c^2\tau\,/\,\eta \tag{1.13}$$

This relationship is the same as the equation of state of a dense gas of solid spheres [5]. It has been confirmed experimentally by A. Ya. Geiler in his thesis for the case of a fluidized bed.

The value of the viscosity μ can be determined by calculating the transport of the horizontal component of momentum from the second layer to the first

$$T_t = N\,(mw_2 - mw_1) = \frac{1}{2}\,\rho_s d c \tau_0{}^{1/3}\,\frac{\tau^{2/3}}{\eta}\,\frac{dw}{dy}$$

where T_t is the tangential stress and y is the coordinate perpendicular to the collision plane. Thus

$$\mu = {}^1/_2\,\rho_s cd\,\tau_0^{1/3}\tau^{2/3}/\,\eta \approx 0.422\ \ \rho_s cd\,\tau^{2/3}\,/\,\eta \tag{1.14}$$

In order to define the parameter γ which occurs in (1.9), we consider the term $\gamma D_{ik}D_{kj}$. Suppose that there is a purely shear plane flow with $dw/dy = 1$. Then

$$D_{ij} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad D_{ik}D_{kj} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

It thus follows that the term $\gamma D_{ik}D_{kj}$ produces normal stresses under the action of a shear. If there

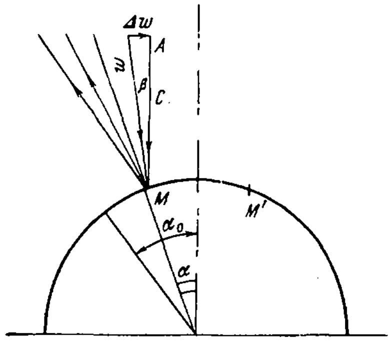

Fig. 1

is relative motion between the layers the spheres will roll over one another and this will lead to a "bursting" force.

In order to make a quantitative estimate of this effect, we consider a hemisphere from the first layer (Fig. 1). In the absence of shear, a particle arriving at the point M moves along the vertical AM with velocity c. If there is a shear velocity Δw, the particle trajectory deviates from the vertical by an angle β, but the vertical component of velocity retains its value so that $w \cos \beta = c$. The vertical momentum transmitted by the particle to the hemisphere

$$i_\beta = mw\,[\cos\beta + \cos(2\alpha - \beta)] = mc\left[1 + \frac{\cos(2\alpha - \beta)}{\cos\beta}\right]$$

We now calculate the excess momentum caused by the shear

$$i_\beta - i_0 = mc \sin 2\alpha \operatorname{tg} \beta$$

At the symmetrical point M', which is characterized by the angle $-\alpha$, the result of the shear is equal in magnitude but opposite in sign. The numbers of particles arriving at the points M and M' in unit time are different in the presence of shear. For the point M the number is proportional to $\cos(\alpha - \beta)$ and for the point M', to $\cos(\alpha + \beta)$. The total excess momentum for M and M' is proportional to the difference $\cos(\alpha - \beta) - \cos(\alpha + \beta) = 2 \sin\alpha \cdot \sin\beta$. Thus for unit area we get

$$\Delta p_s = \frac{Nmc}{2\sin\alpha_0}\int\limits_0^{\alpha_0} \sin 2\alpha \operatorname{tg}\beta\, 2\sin\alpha\sin\beta\cos\alpha\, d\alpha = \frac{p_s}{8}\,\frac{\alpha_0 - {}^1\!/_4 \sin 4\alpha_0}{\sin\alpha_0}\operatorname{tg}\beta\sin\beta$$

where α_0 is the limiting value of α defined by

$$\sin\alpha_0 = \frac{l + d}{2d} = \frac{1}{2}\left(\frac{\tau_0}{\tau}\right)^{1/3} \tag{1.15}$$

Now

$$\sin\beta \approx \operatorname{tg}\beta = \frac{\Delta w}{c} = \frac{1}{c}\,\frac{dw}{dy}(l + d)$$

Assuming τ to be close to τ_0 and using (1.15), we get $\alpha_0 = \pi/6 + \chi$, where χ is a small parameter. Limiting ourselves to the first powers in the expansion, we have

$$\Delta p_s \approx \frac{3}{32}\, p_s \left(\frac{\tau_0}{\tau}\right)^{2/3}\left\{\arcsin\left[\frac{1}{2}\left(\frac{\tau_0}{\tau}\right)^{2/3}\right] - \frac{1}{4\sqrt{3}} - \frac{\pi}{18}\right\}\frac{d^2}{c^2}\left(\frac{dw}{dy}\right)^2$$

whence

$$\gamma \approx 0.019\, p_s\, d^2 / c^2 \tag{1.16}$$

The value of $\gamma > 0$. This corresponds to the action of a "bursting" force in the same direction as the pressure. It can be seen from (1.16) that the term $\gamma D_{ik} D_{kj}$ becomes comparable with p_s when $dw/dy \gg c/d$, i.e., only under extremely nonequilibrium conditions.

2. In order to set up the energy balance in some particular volume, we have to remember that the internal energy of random motion is generated not only from dissipation of the average motion, but also as a result of the Magnus forces which are of an internal nature. There is also a decrease of the internal energy as a result of conversion into thermal energy through friction of the particles with the fluid and through inelastic interparticle collision. The energy balance equation must also allow for energy transfer through mechanical "thermal conductivity."

We define the total energy of the solid phase moving in a volume V as the sum of the kinetic K and internal U energies:

$$K = \frac{1}{2}\int\limits_V \rho_s \tau w^2 dV, \quad U = \int\limits_V \rho_s \tau E\, dV$$

where E is the internal energy per unit mass.

We now apply to the volume V the law of conservation of energy of the solid phase; the change in the total energy of the system must be equal to the work done by external and internal forces plus the energy applied to the system or derived from it. Thus

$$\frac{d(K+U)}{dt} = \int_V (F_i w_i + \rho_s \tau g_i w_i + Q_M - Q_D)\, dV + \int_S (T_{ij} w_j - q_i)\, dS_i \tag{2.1}$$

Here Q_D and Q_M are the powers of the dissipation and Magnus forces per unit volume and q_i is the energy flux vector for the random motion.

If we convolve (1.3) with w_i and integrate over the volume V, then after some standard transformations we get

$$\frac{dK}{dt} = \int_V \left(F_i w_i + \rho_s \tau g_i w_i - \frac{1}{2} T_{ij} D_{ij} \right) dV + \int_S T_{ij} w_j\, dS_i$$

The energy equation (2.1) becomes

$$\frac{dU}{dt} = \int_V \left(Q_M - Q_D + \frac{1}{2} T_{ij} D_{ij} \right) dV - \int_S q_i\, dS_i \tag{2.2}$$

We now transform the surface integral to a volume integral and use the equation

$$\frac{d}{dt} \int_V \rho_s E\, dV = \int_V \rho_s \frac{dE}{dt}\, dV$$

(the volume V is arbitrary) and get from (2.2) that

$$\rho_s \tau \frac{dE}{dt} = Q_M - Q_D + \frac{1}{2} T_{ij} D_{ij} - \frac{\partial q_i}{\partial x_i} \tag{2.3}$$

The internal energy H of one particle consists of the energy of three translational and three rotational degrees of freedom. The vertical component of the random velocity has been taken as c and so the energy in this degree of freedom is $\frac{1}{2}\, mc^2$. For a conservative system in equilibrium, the energy is equally distributed over the degrees of freedom [6]. We thus take $H = (3 + 3)\, mc^2/2 = 3\, mc^2$, whence $E = H/m = 3\, c^2$. From equipartition of energy it follows that

$$mc^2 = {}^1/_{10}\, md^2 \omega^2, \qquad \omega = \sqrt{10}\, c/d \tag{2.4}$$

where ω is the rms component of the angular velocity of a particle.

We now estimate the power of the Magnus forces Q_M. The Magnus force acting on one particle is defined by the last term in (1.5), and so $Q_M = 2\rho\tau\, [\mathbf{u}, \boldsymbol{\omega}]_i c_i$. The value of Q_M depends on the relative orientation of the vectors u_i, ω_i, and c_i. For a rough estimate of the effect, we take the angles between u_i and ω_i and between $[\mathbf{u}, \boldsymbol{\omega}]_i$ and c_i as equal to $\pi/4$. If we take the length of the vector c_i as $\sqrt{3c}$ we can use (2.4) to get

$$Q_M = \sqrt{30}\, \rho\tau c^2\, u / d$$

The coefficient $\sqrt{30}$ is only an estimate and it should be subsequently refined, for example, experimentally.

The power of the internal energy dissipation Q_D is made up of several parts. It is shown in [2] that dissipation over a mean free path can be neglected. The main dissipative processes are connected with collision effects.

Consider first the friction during a collision. Suppose that a collision occurs between two spheres with different velocities moving in opposite directions. The spheres are supposed to be nonrotating before the collision. We suppose that in the collision process the slip of the spheres past one another ceases altogether. As in the case of the impact of a sphere against a fixed plane [7], we have the relationships

$$w_{t1} = {}^5/_7\, w_{t0}, \qquad \omega_1 = {}^{10}/_7\, d^{-1} w_{t0}$$

where w_t is the tangential component of velocity of each sphere; the suffix 0 refers to the state before the collision and the suffix 1 to the state after. If the impact is elastic, then $w_{n1} = -w_{n0}$. The initial kinetic energy of each sphere $h_0 = \frac{1}{2}\, m\,(w_{t0}^2 + w_{n0}^2)$, and after the collision

$$h_1 = \frac{m}{2}(w_{t1}^2 + w_{n1}^2) + \frac{I\omega_1^2}{2} = \frac{5}{14} m w_{t0}^2 + \frac{1}{2} m w_{n0}^2$$

If we take $w_{t0} = c$, then each sphere loses an energy $mc^2/7$ in the collision and an amount $\frac{2}{7}\, mc^2$ is converted into heat. The time between collisions of a single sphere is $2l/c\sqrt{3}$. The sphere thus undergoes $c\sqrt{3}/2l$ collisions per unit time. The total number of collisions per unit time per unit volume is

$$Z = \frac{1}{2}\,\frac{c\sqrt{3}}{2l}\,\frac{6\tau}{\pi d^3} = \frac{3\sqrt{3}}{2}\cdot\frac{c\tau}{\pi d^3 l}$$

If the dissipation is caused only by impact friction then

$$Q_D = \frac{\sqrt{3}}{14}\,\rho_s\,\frac{c^2}{d}\,\frac{\tau}{(\tau_0/\tau)^{1/3} - 1}$$

If the spheres are smooth and inelastic,

$$\Delta h = (1 - k^2)\, mc^2$$

where k is the coefficient of restitution and the dissipative losses per unit volume are

$$Q_D = \frac{\sqrt{3}}{4}(1 - k^2)\,\rho_s\,\frac{c^3}{d}\,\frac{\tau}{(\tau_0/\tau)^{1/3} - 1}$$

In the general case we can take

$$Q_D = \sigma\rho_s\,\frac{c^3}{d}\,\frac{\tau}{(\tau_0/\tau)^{1/3} - 1} \tag{2.5}$$

where to the first approximation

$$\sigma = \frac{\sqrt{3}}{2}\left(\frac{1}{7} + \frac{1 - k^2}{2}\right)$$

This parameter can be considered as a characteristic of the material of the spheres which must be determined by experiment. A more detailed consideration leads to a weak dependence on the concentration τ. The hydrodynamic perturbations caused by the particle collisions also make a contribution to Q_D. When two particles approach each other they communicate kinetic energy to the mass of the liquid between them. This energy is dissipated. The problem of the collision of two spheres is difficult to solve even for the case of an ideal fluid. The loss of energy must be of the order of ρc^2 (with a coefficient of the order of unity). The ratio of the hydrodynamic to the impact dissipation is therefore of the order of ρ/ρ_s for a system of solid particles suspended in a gas, and so the hydrodynamic dissipation can be neglected.

The term $(1/2)T_{ij}D_{ij}$ in (2.3) characterizes the work done in expanding the gas of solid spheres and the dissipation of the energy of average motion. The last term in (2.3) corresponds to thermal conductivity.

Suppose that the layers of spheres have different values of $H = 3mc^2$. Then the transfer of this quantity from layer to layer is given by

$$q = N(H_1 - H_2) = -N\frac{dH}{dy}(l + d)$$

whence

$$q_i = -L\frac{\partial E}{\partial x_i}, \qquad L = \mu \approx 0.422\rho_s c d\tau^{2/3}/\eta \tag{2.6}$$

and the Prandtl number $\Pr = \mu/L = 1$.

We have thus obtained a closed system of equations which describe the behavior of concentrated dispersed systems. If the fluid phase is incompressible, then τ, v_i, p, w_i, p_s, and c are unknowns. They can be determined from (1.1)-(1.4), (1.13), and (2.3). For a compressible fluid phase the system must be supplemented by the equations of state and energy for the fluid phase. For an ideal gas these equations are

$$p = \rho RT, \quad \rho\varepsilon c_p \frac{dT}{dt} = v_i \frac{\partial p}{\partial x_i} + v_i F_i + \frac{\partial}{\partial x_i}\left(\lambda \frac{\partial T}{\partial x_i}\right) + Q \tag{2.7}$$

where R is the universal gas constant, c_p is the specific heat at constant pressure, T is the absolute temperature, λ is the thermal conductivity of the gas, and Q is the heat obtained by the gas from the solid phase. Using (1.6) and (1.7), we can represent the second equation in (2.7) as

$$\varepsilon\rho \frac{d}{dt}\left(c_p T + \frac{v_i^2}{2}\right) = \frac{\partial}{\partial x_i}\left(\lambda \frac{\partial T}{\partial x_i}\right) \tag{2.8}$$

If thermal processes are important in the dispersed system (for example if an amount of heat Q_p is produced in the solid phase), then the thermal balance equation must be written down for the solid phase

$$\rho_s \tau c_s \frac{dT_s}{dt} = Q_p - Q + \frac{\partial}{\partial x_i}\left(\Lambda \frac{\partial T_s}{\partial x_i}\right) \tag{2.9}$$

where c_s and T_s are the thermal capacity and temperature of the solid phase, Λ is the diffusion conductivity (which is quite important in a number of cases [8]). The quantity Q_p includes the dissipative heat Q_D. For convective thermal exchange

$$Q = 6\,\tau d^{-1}\alpha\,(T_s - T)$$

where α is the heat transfer coefficient.

3. The boundary conditions for the parameters characterizing the fluid phase are formulated in the standard way for hydrodynamics.

The boundary conditions for the solid phase are determined by the perturbations experienced by a solid surface S placed in the particle stream. On this surface S we have the condition that no particles can penetrate, i.e., $w_n|_S = W_n$, where W_i is the velocity of motion of an element of S; if the surface is stationary then $w_n|_S = 0$. The motion of the particles along a wall can take place with a significant slip velocity θ. The particles acquire significant angular velocities from collisions with the wall and also translational velocities relative to the layer. This produces a loss of a considerable fraction of the tangential momentum by the particles and increases the internal energy of the layer.

The wall has an ordering effect on the adjacent layer of particles and causes local changes in the porosity. However, this effect is outside the scope of the present paper, which is based on the assumption that there is total disorder in the particle positions.

We now consider the impact of a particle on the wall, assuming this impact to be elastic. Since the angular velocity of the particles inside the layer is on average equal to zero, we take the spheres to be nonrotating before the collision. The loss in the tangential momentum of a particle is then [7]

$$m\Delta w_t = \begin{cases} {}^2/_7 m\theta & (\theta/7c \leqslant f) \\ 2fmc & (\theta/7c \geqslant f) \end{cases} \tag{3.1}$$

where f is the coefficient of sliding friction at the surface; the case $\theta/7c \leq f$ corresponds to the cessation of slip. For the angular velocity we have

$$\omega d = \begin{cases} {}^{10}/_7 \theta & (\theta/7c \leqslant f) \\ 2fmc & (\theta/7c \geqslant f) \end{cases} \tag{3.2}$$

Multiplying (3.1) by N, we get the tangential stress at the wall

$$T_t = \begin{cases} {}^1/_7\theta c^{-1} p_s & (\theta / 7c \leqslant f) \\ f p_s & (\theta / 7c \geqslant f) \end{cases} \tag{3.3}$$

The tangential stress varies continuously with the quantity $\theta/7c$. When $\theta/7c \leq f$ the stress depends linearly on θ as is characteristic of fluid friction. When $\theta/7c \geq f$ the stress becomes constant, i.e., the dry friction law holds. The discontinuity in (3.3) is due to the elementary treatment which assumes that the quantity $\theta/7c$ is the same for all spheres and that the initial angular velocity is zero. If the particle distributions in linear and angular velocity are taken to be Maxwellian (for example) the relationship becomes a smooth one.

The derivation of (3.3) ignored the velocity gradient in the layer which, when the term $\gamma D_{ik} D_{kj}$ is included in (1.9), produces an additional normal stress. In general (3.3) should be replaced by

$$T_t = \xi T_n, \quad \xi = \begin{cases} \theta / 7c & (\theta / 7c \leqslant f) \\ f & (\theta / 7c \geqslant f) \end{cases} \tag{3.4}$$

Here T_n is the total normal stress at the wall. The boundary condition for the tangential component of velocity (allowing for the particular law of friction at the surface) can be found by equating the quantity T_t from (3.4) to the tangential stress derived from (1.9).

The boundary conditions for the internal energy E depend on the properties of the particles and the wall. The wall might, for example, generate energy by vibration, or absorb it.

An additional energy flux, connected with the conversion of part of the energy of ordered motion at the wall into random motion, can be found in the form

$$q_c = \left(\frac{m \Delta w_t^2}{2} + \frac{I\omega^2}{2}\right) N = \frac{7}{2} \rho_s \frac{\tau}{\eta} \xi^2 c^3 \tag{3.5}$$

The boundary conditions for the energy equation including (3.5) in the absence of other processes at the wall can be found by equating the right sides of (3.5) and (2.6).

The power of the surface forces $\int_S T_{ij} w_j dS_i$, which is included in (2.1), does not affect the internal energy balance as it does not occur in (2.2).

4. The particular case is considered in [2] of a dispersed system in which the solid phase has no average motion. Since $D_{ij} = 0$, the equation of motion of the solid phase simplifies and reduces to the condition for hydrostatic equilibrium. The energy equation reduces to $Q_M = Q_D$. A particular property of this type of system (a model of a fluidized bed) is the liquid–vapor phase transition.

We consider now another example – the analog of a plane Couette flow for a granulated material in which there is no fluid phase ($\rho = 0$). When a viscous gas flows between two parallel walls which are moving with the same speed in opposite directions the frictional heat must be taken away through the walls in order for stationary conditions to exist. Because of the presence of a sink of internal energy in the form of the dissipated power Q_D, adiabatic conditions are possible ($q|_S = 0$), and these are considered below.

We look for solutions with constant values for the velocity gradient $dw/dy = a$, the internal energy E, and the pressure p_s. Equation (1.9) gives

$$T_{ij} = -(p_s + \gamma a)^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \mu a \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

It thus follows that

$$T_n = p_s + \gamma a^2, \quad T_t = \mu a \tag{4.1}$$

Since $\frac{1}{2} T_{ij}D_{ij} = \mu a^2$, the energy equation (2.3) reduces to $Q_D = \mu a^2$. Using (2.5) and (1.14), we get

$$a^2d^2 = \frac{2\sigma}{\tau_0^{2/3}} c^2\tau^{2/3}, \quad ad = \frac{\sqrt{2\sigma}}{\tau_0^{1/3}} c\tau^{1/3} \tag{4.2}$$

We now use (3.4), substituting (4.1) into it. With (1.13), (1.14), and (1.16), we get

$$\xi = \sqrt{\frac{\sigma}{2}}\left[1 + 0.038\sigma\left(\frac{\tau}{\tau_0}\right)^{1/3}\right] \approx \sqrt{\frac{\sigma}{2}} \tag{4.3}$$

For elastic rough spheres we have $\sigma = \sqrt{3/14}$, $\xi = 0.25$.

Thus for this Couette flow ξ is a constant number. If the coefficient of friction $f > \xi$, then from (3.4) $\theta/7c = \xi$. and this expression defines the jump in velocity at the flow boundary. A gradient flow is impossible for $f < \xi$. When there is no external force field a normal pressure is produced only by a "thermal" expansion of the layer, and it is possible to get the trivial solution $c = 0$, $T_{ij} = 0$, $a = 0$. The planes freely slide over the particle layer without any interaction. A nontrivial solution exists only for $f > \xi$. Then $\theta = 7\xi c$. Suppose that the relative velocity of the plates is given as $w_0 = ab + 2\xi c$, where b is the width of the layer. Solving this equation together with (4.2) and using (4.3), we get

$$c = \frac{w_0}{2\xi\Delta}, \quad a = \frac{dw}{dy} = \left(\frac{\tau}{\tau_0}\right)^{1/3} \frac{w_0}{d\Delta} \quad \left(\Delta = 1 + \left(\frac{\tau_0}{\tau}\right)^{1/3} \frac{b}{d}\right)$$

The quantity τ in this problem must be given. The tangential stress arising at the plates as they move is

$$T_t = \frac{\rho_T}{4\xi} \frac{\tau}{\eta} \frac{w_0^2}{\Delta^2}$$

The results which have been obtained are in qualitative agreement with experiment. Suppose that a granulated material is placed between two cylinders of which the inside one rotates and the outer one is fixed. If the internal cylinder is smooth, the layer does not affect the rotation. The motion of the granulated material begins only when the inside cylinder becomes sufficiently rough.

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